A cohomology theory for (infinitesimally) multiplicative tensors

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Abstract

Multiplicative tensors are sections of certain vector bundles compatible with a multiplication of a Lie group(oid). Similarly, there is a corresponding infinitesimal notion for Lie algebroids. We describe and discuss properties of a general type of cohomology theory in which (infinitesimally) multiplicative tensors appear as cocycles.

Notation

We write

$$\text{VB-groupoid: } \begin{tikzcd} V_G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & V_M \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \qquad \text{VB-algebroid: } \begin{tikzcd} V_A \arrow[r, Rightarrow] \arrow[d] & V_M \arrow[d] \\ A \arrow[r, Rightarrow] & M \end{tikzcd}$$

(C_M for the core) horizontal = Lie groupoid/algebroid, vertical = vector bundle.

Tensor products of VB-groupoids

Tensor product of composable arrows of $V_{G,1}, \ldots, V_{G,n}$ form simplicial vector bundle:

$$\begin{tikzcd} \bigotimes_{r=1}^n V_{G,r}^{(2)} \arrow[r, shift left=3] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=3, leftarrow] \arrow[d] & \bigotimes_{r=1}^n V_{G,r} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d] & \bigotimes_{r=1}^n V_{M,r} \arrow[d] \\ G^{(2)} \arrow[r, shift left=3] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=3, leftarrow] & G \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & M \end{tikzcd}$$

(face maps and degeneracy maps defined "componentwise"). Main definition:

Definition The linear (simplicial) complex of the tensor product of $V_{G,1}, \ldots, V_{G,n}$ is

$$C^\bullet_{\mathrm{lin}}(\textstyle\bigotimes_r V_{G,r}) = C^\infty_{\mathrm{lin}}(\textstyle\bigotimes_r V_{G,r}^{(\bullet)}) \\ = \Gamma(\textstyle\bigotimes_r (V_{G,r}^{(\bullet)})^*).$$

Multiplicative tensors appear as 1-cocycles:

- $V_{G,1} = V_{G,2} = TG$: multiplicative symplectic forms.
- $V_{G,1} = V_{G,2} = T^*G$: multiplicative Poisson tensors.
- $V_{G,1} = TG$, $V_{G,2} = T^*G$: multiplicative complex structures.
- $V_{G,1} = TG$, $V_{G,2} = \mathbf{s}^*\mathfrak{k}^*$ with $\mathfrak{k}$ a bundle of ideals: multiplicative Ehresmann connections.

The (sub)complexes are therefore relevant for existence results and deformation theory.

Tensor products of VB-algebroids

Tensor product of $V_{A,1}, \ldots, V_{A,n}$ in graded geometry (so as differential graded ΩA-modules):

$$\textstyle\bigotimes_{r=1}^n V_{A,r}[1] = \bigotimes_{r,\Omega A} \Omega_{\mathrm{lin}}(V_{A,r}).$$

Elements are determined on linear sections. So, consider:

$$\widehat{V}_A = \widehat{V}_{A,1} \times_A \cdots \times_A \widehat{V}_{A,n} \Rightarrow M$$

with $\widehat{V}_{A,r} \Rightarrow M$ the fat algebroids:

$$\Gamma \widehat{V}_{A,r} = \Gamma_{\mathrm{lin}}(V_{M,r}, V_{A,r}).$$

There is a short exact sequence

$$0 \longrightarrow \textstyle\bigoplus_{r=1}^n \mathrm{Hom}(V_{M,r}, C_{M,r}) \longrightarrow \widehat{V}_A \longrightarrow A \longrightarrow 0$$

and $\widehat{V}_A$ is represented on the tensor product of the cochain complexes

$$V_{M,r}^* \xrightarrow{-\rho^*} C_{M,r}^*.$$

We write

$$\Omega^\bullet(\widehat{V}_A; \textstyle\bigotimes) = \Omega^\bullet(\widehat{V}_A; \bigotimes_r V_{M,r}^*) \oplus \cdots \oplus \Omega^{\bullet - n}(\widehat{V}_A; \bigotimes_r C_{M,r}^*).$$

The following forms a subcomplex of $\Omega^\bullet(\widehat{V}_A; \bigotimes_r V_{M,r}^*)$ isomorphic to $\bigotimes_r \Omega_{\mathrm{lin}}(V_{A,r})$.

Definition Cochains of "Weil-type": $\omega \in \Omega^\bullet(\widehat{V}_A; \bigotimes_r V_{M,r}^*)$ such that $\exists (\omega_{n'}) \in \Omega^\bullet(\widehat{V}_A; \bigotimes)$ with $\omega = \omega_0$ and

$$\iota_\varphi \omega_{n'} = \omega_{n'+1} \circ \varphi \quad \forall \varphi \in \mathrm{Hom}(V_{M,r}, C_{M,r}).$$

Similar to before, infinitesimally multiplicative tensors appear as 1-cocycles.

Relation to other models

We clarify the relation to other models, in particular, those given in the references.

Van Est map

VB-groupoids $V_{G,1}, \ldots, V_{G,n} \overset{\text{Lie}}{\rightsquigarrow}$ VB-algebroids $V_{A,1}, \ldots, V_{A,n}$: Van Est map

$$\mathrm{VE} : C_{\mathrm{lin}}(\textstyle\bigotimes_r V_{G,r}) \to \bigotimes_r \Omega_{\mathrm{lin}}(V_{A,r}).$$

Theorem If G is $\leq n$-connected, VE is an isomorphism in cohomology in degrees $\leq n$.

Morita Invariance

Bibundle approach to VB-Morita equivalences: a Morita bibundle

$$\begin{tikzcd} V_G \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & V_P \arrow[dll, "\tau"] \arrow[drr, "\sigma"'] & \circlearrowleft & V_H \arrow[d, shift left] \arrow[d, shift right] \\ V_M & & & & V_N \end{tikzcd}$$

with $V_P \to P$ a vector bundle, basemaps form Morita bibundle $G \circlearrowright P \circlearrowleft H$ and actions

$$V_G \times_{V_M} V_P \to V_P \text{ and } V_P \times_{V_N} V_H \to V_P$$

are linear. We prove:

Theorem Given $V_{G,r} \circlearrowright V_{P,r} \circlearrowleft V_{H,r}$ over $G \circlearrowright P \circlearrowleft H$, there is a quasi-isomorphism

$$C_{\mathrm{lin}}(\textstyle\bigotimes_r V_{G,r}) \to C_{\mathrm{lin}}(\bigotimes_r V_{H,r}).$$

Relation to representation up to homotopy (ruth)

VB-groupoids $V_{G,1}, \ldots, V_{G,n} \rightsquigarrow$ 2-term ruths $\mathcal{E}_{G,1}, \ldots, \mathcal{E}_{G,n}$ over

$$V_{M,r}^* \xrightarrow{-\mathbf{t}^*} C_{M,r}^*.$$

Tensor product $\bigotimes_r \mathcal{E}_{G,r}$ of ruths is ruth over tensor product of $V_{M,r}^* \to C_{M,r}^*$. We prove:

Theorem There is a quasi-isomorphism

$$C_{\mathrm{lin}}(\textstyle\bigotimes_r V_{G,r}) \to \bigotimes_r \mathcal{E}_{G,r}.$$

In particular:

Theorem If G is proper, the cohomology of $C^\bullet_{\mathrm{lin}}(\bigotimes_r V_{G,r})$ vanishes in degrees $\bullet > n$.

Remarks and recent work in progress

Remarks

- All quasi-isomorphisms proved using the perturbation lemma of a double complex.
- In particular: explicit expressions for quasi-isomorphisms and homotopy data.
- We discuss cup products. The Van Est map respects the products.
- In Morita invariance: new proof that $C_{\mathrm{VB}} V_G \hookrightarrow C_{\mathrm{lin}} V_G$ is a quasi-isomorphism.
- Also: new definition of Morita equivalence between ruths.

Recent work in progress

- Global analogue of the fat algebroid is the fat groupoid:
$$\widehat{V}_G = \widehat{V}_{G,1} \times_G \cdots \times_G \widehat{V}_{G,n}$$
$$\widehat{V}_{G,r} = \{H_g : (V_{M,r})_{\mathbf{s}g} \to (V_{G,r})_g \mid \mathbf{s}H_g = 1 \text{ and } \mathbf{t}H_g \text{ a linear isomorphism}\}.$$
There is a natural global candidate for "Weil-type" cochain subcomplex; we investigate the relation to the linear complex and ruths.
- In particular, we proved: $C_{\mathrm{VB}} V_G$ is isomorphic to this subcomplex of $C(\widehat{V}_G; V_M^*)$. If $V_G = T^*G$, then $\widehat{V}_G \cong J^1 G$: reinterpretation of the deformation complex. Relevant for the deformation theory of groupoids; this will be the content of a seperate work.
- Existence of (infinitesimally) multiplicative connections on VB-groupoids (VB-algebroids) governed by the Atiyah class. We expect: classes are related by the Van Est map.

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